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SYNCHRONIZATION OF PENDULUM CLOCKS SUSPENDED ON AN ELASTIC BEAM

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A synchronous regime in the Huygens problem is studied with allowance for nonlinear interaction between a beam and pendulums. It is shown that the precise out-of-phase motion of different pendulums noted by Huygens cannot occur. The case where the motion of the pendulums is synchronous and close to the out-of-phase motion is studied.

Key words: *synchronization, parametric excitation, harmonic-balance method.*

Rate synchronization of two pendulum clocks suspended on an elastic beam was first noted by Huygens [1]. This phenomenon has been studied in detail [2–5]; however, the problem was formulated under the assumption of linear interaction between the beam and pendulums. The pendulum oscillations are sustained by clock mechanisms represented approximately by variable damping of the Van der Pol type.

In the present paper, we consider nonlinear interaction between the beam and clocks also modeled by the Van der Pol systems. Beam oscillations lead to parametric excitation of pendulum oscillations due to vibration of suspension points. The maximum excitation corresponds to the principal parametric-resonance zones where the rate of the clocks is synchronized with a frequency equal to half of the beam-oscillation frequency. Thus, it is assumed that synchronization occurs by capturing the frequency of two Van der Pol systems under parametric excitation. Pendulum oscillations, in turn, sustain oscillations of the beam by means of centrifugal forces for which the frequency of the pendulums increases twofold.

1. Let the clocks be suspended symmetrically about the middle of the beam (Fig. 1). The beam deflection is described by the vertical displacement x of one of the suspension points for a specified symmetric deflection function. Positions of the pendulums are determined by the angles φ_i $(i = 1, 2)$ counted from the vertical axes x_i in the opposite directions and assumed to be small. We assume that the pendulum masses m are identical, whereas the lengths l_i are close but still different.

We write the kinetic energy of the system

$$T = \frac{1}{2}\,M\dot{x}^2 + \frac{1}{2}\,m\sum_{i=1}^{2}(\dot{x}^2 - 2l_i\sin(\varphi_i\,\dot{x}\dot{\varphi}_i) + l_i^2\dot{\varphi}_i^2),$$

where M is the effective mass of the beam with the virtual masses of clock mechanisms and cases (without pendulums). The potential energy of the system is

$$\Pi = \frac{1}{2}\,M\omega_{\rm beam}^2 x^2 - mg\sum_{i=1}^{2} l_i\cos\varphi_i,$$

where $\omega_{\rm beam}$ is the frequency of free oscillations of the beam with virtual masses without pendulums and g is the acceleration of gravity. The equations of motion have the form

$$\begin{gathered}(M+2m)\ddot{x} - m\sum_{i=1}^{2} l_i(\varphi_i\ddot{\varphi}_i + \dot{\varphi}_i^2) = -M\omega_{\rm beam}^2 x + Q_x,\\ ml_i^2\ddot{\varphi}_i - ml_i\varphi_i\ddot{x} = -mgl_i\varphi_i + Q_i, \qquad i = 1, 2.\end{gathered} \tag{1}$$

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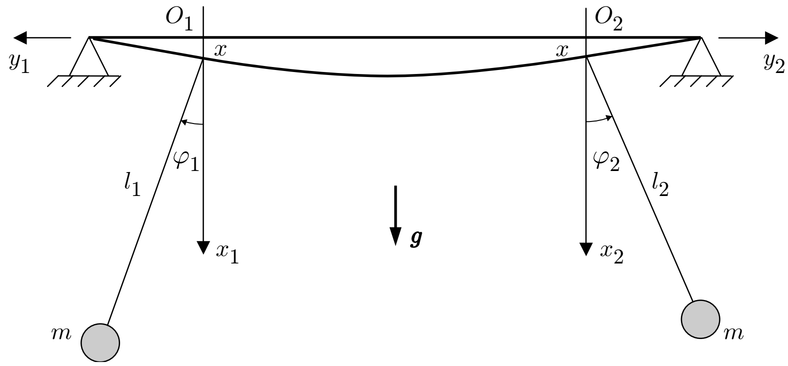

Fig. 1

Here Q_x is the linear-friction force in the beam and Q_i are the Van der Pol nonlinear-friction forces. We write these generalized forces as

$$\frac{Q_x}{M+2m} = -\varepsilon_0 \dot{x}, \qquad \frac{Q_i}{l_i m} = \varepsilon_i \Big(1 - \frac{S_i^2}{S_0^2}\Big) \dot{S}_i,$$

where ε_0 and ε_i are the friction coefficients, $S_i = l_i \varphi_i$, and S_0 are the deflections of the pendulums at which the damping changes its sign.

We write Eqs. (1) as

$$x'' + 2\,\frac{\varepsilon_0}{\omega}\,x' + 4\,\frac{\omega_0^2}{\omega^2}\,x = \mu[(y_1^2)'' + (y_2^2)''], \qquad y_i'' + \Big(a_i - \frac{x''}{l_i}\Big) y_i = \delta(1 - y_i^2) y_i', \tag{2}$$

where

$$\frac{M}{M+2m}\,\omega_{\text{beam}}^2 = \omega_0^2, \qquad \frac{mS_0^2}{2l_1(M+2m)} \approx \frac{mS_0^2}{2l_2(M+2m)} = \mu,$$

$$\frac{S_i}{S_0} = y_i, \qquad a_i = \frac{4\omega_i^2}{\omega^2}, \qquad \frac{g}{l_i} = \omega_i^2, \qquad 2\,\frac{\varepsilon_1 S_0^2}{\omega l_1} \approx 2\,\frac{\varepsilon_2 S_0^2}{\omega l_2} = \delta.$$

In Eqs. (2), the prime denotes differentiation with respect to the nondimensional time $z = \omega t/2$ (ω is the unknown frequency of beam oscillations). We set $l_1 \approx l_2$ in the coefficients of small nonlinear terms and in the coefficients of the Van der Pol damping.

2. It is expedient to determine the synchronous regime using the harmonic-balance principle. We assume that the beam performs harmonic oscillations according to the law $x = -X\cos 2z$. Substitution of the expression for x into Eq. (2) yields

$$(1 - \omega_0^2/\omega^2) q \cos 2z + \delta_0 q \sin 2z = \nu[(y_1^2)'' + (y_2^2)''];\tag{3}$$

$$y_i'' + (a_i - 2q\cos 2z) y_i = \delta(1 - y_i^2) y', \qquad i = 1, 2.\tag{4}$$

Here $q = 2X/l$, $\delta_0 = \varepsilon_0/\omega$, and $\nu = 32\mu/l$ ($l_1 \approx l_2 = l$).

Equations (4) without the right side are the Mathieu equations. Direct application of the harmonic-balance principle to these equations involves difficulties since the variable coefficients multiplied by harmonic functions of the approximate solution lead to mixed harmonics. To take into account parametric excitation, one has to introduce the mixed harmonics into the approximate solution, which makes the calculations more difficult. Following [6], we first eliminate the term with the variable coefficient from the equation. We seek the solution of Eqs. (4), which describes the steady motion in the region of the expected principal parametric resonance, in the form

$$y_i \approx C_i \operatorname{ce}_1(z, q) + D_i \operatorname{se}_1(z, q).\tag{5}$$

Here $\operatorname{ce}_1(z, q)$ and $\operatorname{se}_1(z, q)$ are the Mathieu functions of the first order and C_i and D_i are unknown constants. Substitution of this solution into Eq. (4) yields

$$(C_i \operatorname{ce}_1(z, q) + D_i \operatorname{se}_1(z, q))'' + (a_i - 2q\cos 2z)(C_i \operatorname{ce}_1(z, q) + D_i \operatorname{se}_1(z, q)) = \delta(1 - y_i^2) y_i'.$$

Bearing in mind that the Mathieu functions of the first order satisfy the Mathieu equation for the eigenvalues $a_c^{(1)}$ and $a_s^{(1)}$, we obtain

$$(a_i - a_c^{(1)}) C_i \operatorname{ce}_1(z,q) + (a_i - a_s^{(1)}) D_i \operatorname{se}_1(z,q) = \delta(1 - y_i^2) y_i'. \tag{6}$$

We represent the Mathieu functions in the last equation and approximate solution (5) by the leading terms of their expansion into power series of q

$$\operatorname{ce}_1(z,q) \approx \cos z, \qquad \operatorname{se}_1(z,q) \approx \sin z$$

and introduce the phase shifts α_i into the pendulum oscillations:

$$y_i = A_i \cos(z - \alpha_i).$$

In (6), equating the coefficients of identical harmonics in accordance with the harmonic-balance principle and taking into account the relations

$$C_i = A_i \cos\alpha_i, \qquad D_i = A_i \sin\alpha_i,$$

we obtain the following system of two equations for each pendulum:

$$(a_i - a_c^1)\cos\alpha_i = \delta(1 - A_i^2/4)\sin\alpha_i, \qquad (a_i - a_s^1)\sin\alpha_i = -\delta(1 - A_i^2/4)\cos\alpha_i. \tag{7}$$

Using the well-known series for the eigenvalues of the Mathieu functions, we assume that

$$a_c^{(1)} = 1 + q - q^2/8 - \ldots \approx 1 + q, \qquad a_s^{(1)} = 1 - q - q^2/8 + \ldots \approx 1 - q.$$

Eliminating α_i from Eqs. (7), we obtain

$$(a_i - 1)^2 - q^2 + \delta^2(1 - A_i^2/4)^2 = 0.$$

It follows that, for small parametric excitation of q and $a_i \approx 1$, the relative amplitudes of pendulum oscillations are close to $A_i \approx 2$.

Substituting the expressions for y_i into Eq. (3) and determining the coefficients of $\cos 2z$ and $\sin 2z$, we obtain

$$(1 - \omega_0^2/\omega^2) q = -\nu(\cos 2\alpha_1 + \cos 2\alpha_2)/2, \qquad \delta_0 q = -\nu(\sin 2\alpha_1 + \sin 2\alpha_2)/2. \tag{8}$$

Thus, the system of equations for the synchronous regime comprises four equations (7) ($i = 1, 2$) and two equations (8).

3. Let us find the Huygens regime $\alpha_1 = \alpha_2$ among possible synchronous regimes. We introduce the notation

$$A_i^2 = 4(1 + \gamma_i), \qquad \omega^2 = \omega_0^2/(1 + \beta), \qquad 4\omega_i^2 = \omega_0^2(1 + \tau_i).$$

Then, $a_i = 1 + \beta + \tau_i + \beta\tau_i = 1 + \Delta_i$. System (7) becomes

$$(\Delta_i - q)\cos\alpha_i = -\delta\gamma_i \sin\alpha_i, \qquad (\Delta_i + q)\sin\alpha_i = \delta\gamma_i\cos\alpha_i. \tag{9}$$

Dividing the first equation of (9) by the second equation for each i and using the formula of the double argument for the cosine, we obtain

$$\cos 2\alpha_i = \Delta_i/q, \qquad \sin 2\alpha_i = -\sqrt{1 - \Delta_i^2/q^2}. \tag{10}$$

For both pendulums, the quantities $\sin 2\alpha_i$ should be negative for the second equation of (8) to be satisfied for reasonably small values of δ_0 and not lead to the degenerate case $\delta_0 = 0$ for motion of identical pendulums.

If the initial phases are rigorously equal, relations (10) imply that $\Delta_1 = \Delta_2$, which corresponds to the trivial case of motion of identical pendulums ($\tau_1 = \tau_2$). Let the small quantity χ characterize the difference in the rates of separate isolated clocks. We introduce the small difference between the initial phases of the pendulums η:

$$\tau_1 = \tau + \chi/2, \quad \alpha_1 = \alpha + \eta/2, \qquad \tau_2 = \tau - \chi/2, \quad \alpha_2 = \alpha - \eta/2.$$

It follows from (10) that

$$\begin{aligned} \cos 2\alpha_1 &= \cos 2\alpha - \eta\sin 2\alpha = \Delta/q + (1+\beta)\chi/(2q), \\ \cos 2\alpha_2 &= \cos 2\alpha + \eta\sin 2\alpha = \Delta/q - (1+\beta)\chi/(2q), \end{aligned} \tag{11}$$

where $\Delta = \beta + \tau + \beta\tau$.

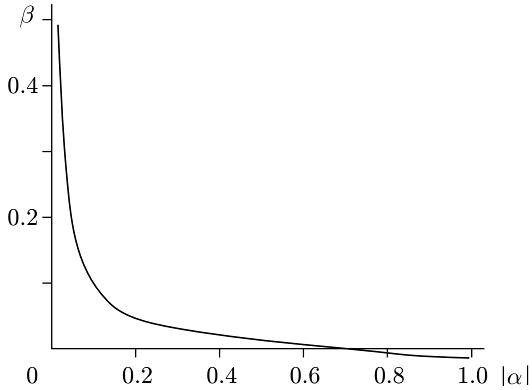

Fig. 2

Expressions (11) yield the relation between χ and η

$$\eta = -(1+\beta)\chi/(2q\sin 2\alpha) \qquad \text{or} \qquad \eta = k(\alpha)\chi \tag{12}$$

and the following expressions for determining the mean value of the initial phases:

$$\cos 2\alpha = \Delta/q, \qquad \sin 2\alpha = -\sqrt{1-\Delta^2/q^2}, \qquad \alpha < 0. \tag{13}$$

With allowance for

$$\sin 2\alpha_1 = \sin 2\alpha + \eta\cos 2\alpha, \qquad \sin 2\alpha_2 = \sin 2\alpha - \eta\cos 2\alpha,$$

from Eqs. (8) we obtain

$$\beta q = \nu\cos 2\alpha, \qquad \delta_0 q = -\nu\sin 2\alpha.$$

This implies that

$$\beta = -\delta_0\cot\, 2\alpha, \qquad q = -(\nu/\delta_0)\sin 2\alpha. \tag{14}$$

To calculate the synchronous regime, we choose a certain value of δ_0. Given the coefficient ε_0, the value of δ_0 can be refined once the oscillation frequency ω is determined. Specifying the mean value of the initial phases α and using relations (14), (13), and (11), we find successively the quantities β, q, Δ, and τ, and the coefficient $k(\alpha)$ from (12). Given particular data, we can assume that the mean values of the eigenfrequencies of the pendulums and the eigenfrequency of the beam, i.e., the quantity $\tau(\alpha)$, are known. Of practical interest is the case $\tau < 0$ ($|\alpha| < \pi/4$) where the eigenfrequency of the beam is more than twice the mean eigenfrequency of the pendulums. Figure 2 shows the calculated dependence $\beta(\alpha)$. One can see that the frequency of beam oscillations does not necessarily coincide with its eigenfrequency. The calculation were performed for $\delta_0 = 0.02$.

4. We estimate the quantities χ and η. Let $T = 86{,}400$ sec be the duration of a day and T_i be the oscillation period of the ith pendulum. For certainty, we assume that $\omega_1 > \omega_2$. In this case, the daily difference in the rate n (difference between the number of pendulum oscillations in a day) is

$$n = T/T_1 - T/T_2 = T(\omega_1-\omega_2)/(2\pi).$$

Taking into account the relation

$$\omega_i = \frac{\omega_0}{2}\,(1+\tau_i)^{1/2} = \frac{\omega_0}{2}\,(1+\tau)^{1/2} \pm \frac{\omega_0}{4}\,\frac{\chi}{(1+\tau)^{1/2}},$$

we obtain

$$\chi = (1+\tau)nT_{\text{mean}}/T,$$

where T_{mean} is the mean period of pendulum oscillations. According to the data of Huygens [1], $T_{\text{mean}} \approx 1$ sec. Confining ourselves to nonpositive values of τ, we find that $\max\chi \approx 1.15\cdot 10^{-5}$ for $n = 1$.

Huygens [1] considered two ship chronometers placed in cases that contained approximately 100 pounds of lead. Owing to these masses, the eigenfrequency of the beam decreases substantially and approaches the doubled frequency of the pendulums. It is worth noting that the chronometers were used to determine the longitudinal location of the ship, and the discrepancy in the clock rate equal to one oscillation of the second pendulum in a day corresponded to the error in determining the longitude, approximately equal to 500 m in a day (at the equator). Thus, the number n cannot exceed several units.

Calculating the coefficient $k(\alpha)$ from formula (12), we obtain $175 < k(\alpha) < 1600$ for $0.175 < |\alpha| < \pi/4$. In this case, the difference between the initial phases of the pendulums lies within the interval $0.002 < \eta < 0.018$ ($n = 1$). If the difference in the clock rate is equal to several oscillations of the pendulums in a day, the difference in the initial phases remains almost unnoticeable.

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